Contraction Semigroups on Hilbert Spaces

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Outline of the talk

We discuss a generalization of mean ergodic theorems for single operators to sets of operators.

Notation

Let us fix some notations.

- X - Banach space
- H - Hilbert space
- $\mathcal{B}(X)$ - the space of all bounded linear operators on X
- $fix(T)$ - the space of all fixed points of T
- $N(T)$ - null space of T
- $R(T)$ - range (space) of T
- $\overline{R(T)}$ - closure of $R(T)$

Introduction

The Ergodenhypothese (ergodic hypothesis) was formulated around 1880 by Boltzmann, where he claimed that for an ideal gas, over time goes through every physically feasible state. This used by Boltzmann to deduce that independently of the initial state the average number of visits of a region in the phase space is proportional to the volume of the region. Rephrased and simplified this means time mean equals space mean. However this was doubted to be true by Lord Kelvin and Poincare and disproven by Plancherel and Rosenthal in 1913. Reformulation as quasi ergodic hypothesis which means that the for almost every initial value this property holds. This was finally be proven independently by von Neumann (mean ergodic theorem) and by Birkhoff (pointwise ergodic theorem).

Proofs of which were published nearly simultaneously in PNAS in 1931 and 1932. These theorems were of great significance both in mathematics and in statistical mechanics.

Introduction

In statistical mechanics they provided a key insight into a 60-y-old fundamental problem of the subject – namely, the rationale for the hypothesis that time averages can be set equal to phase averages. The evolution of this problem is traced from the origins of statistical mechanics and Boltzman's ergodic hypothesis to the Ehrenfests' quasi-ergodic hypothesis, and then to the ergodic theorems. These ergodic theorems initiated a new field of mathematical-research called ergodic theory that has thrived ever since. George D. Birkhoff and John von Neumann published separate and virtually simultaneous path-breaking papers in which the two authors proved slightly different versions of what came to be known (as a result of these papers) as the ergodic theorem.

Introduction

The techniques that they used were strikingly different, but they arrived at very similar results. The ergodic theorem, when applied say to a mechanical system such as one might meet in statistical mechanics or in celestial mechanics, allows one to conclude remarkable results about the average behavior of the system over long periods of time, provided that the system is metrically transitive (a concept to be defined below). First of all, these two papers provided a key insight into a 60-y-old fundamental problem of statistical mechanics, namely the rationale for the hypothesis that time averages can be set equal to phase averages, but also initiated a new field of mathematical research called ergodic theory, which has thrived for more than 80 y. Subsequent research in ergodic theory since 1932 has further expanded the connection between the ergodic theorem and this core hypothesis of statistical mechanics.

Koopman Operator

We shall start with the von Neumann mean ergodic theorem using a little bit operator theory.

Let (X, ϕ) be a measure-preserving system. Since $\phi : X \to X$ is only measure preserving and measurable it might be nonlinear, therefore we want to associate a suitable linear operator T on the measurable functions by

$$f \mapsto Tf := f \circ \phi.$$

This operator $T = T_\phi$ is called the **Koopman operator** in the literature. Clearly Tf is again a measurable function, it is linear and we can rewrite $f \circ \phi^n = T^n f$.

Note that the time mean of a function $f : X \to \mathbb{R}$ under the first $n \in \mathbb{N}$ iterates of $T := (f \longmapsto f \circ \phi)$ is given by

$$A_n f = \frac{1}{n}(f + f \circ \phi + \cdots + f \circ \phi^{n-1}) = \frac{1}{n} \sum_{j=0}^{n-1} T^j f.$$

Von Neumann's theorem

Von Neumann's theorem deals with these averages $A_n f$ for f from the Hilbert space $L^2(X)$.

Theorem 1 (Von Neumann).

Let (X, ϕ) be a measure-preserving system and consider the Koopman operator $T := T_\phi$. For each $f \in L^2(X)$ the limt

$$\lim_{n\to\infty} A_n f = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f$$

exists in the L^2-sense and is a fixed point of T.

We shall not discuss von Neumannn's original proof, but we will prove the von Neumann theorem in a more general form.

Cesaro averages

For a linear operator T on a vector space X we let

$$A_n[T] := \frac{1}{n} \sum_{j=0}^{n-1} T^j \quad (n \in \mathbb{N})$$

be the **Cesaro averages** of the first n iterates of T.

We denote by

$$fix(T) := \{f \in X : Tf = f\} = N(I - T)$$

the **fixed space** of T.

Lemma 2.

Let X be a Banach space, $T \in \mathcal{B}(X)$ and $A_n := A_n[T]$. Then

- *If $f \in fix(T)$, then $A_n f = f$ for all $n \in \mathbb{N}$, and hence $A_n f \to f$;*
- *If $A_n f \to g$, then $Tg = g$ and $A_n Tf \to g$;*

  [Hint : for $n \in \mathbb{N}$, $A_n T = TA_n = \frac{n+1}{n} A_{n+1} - \frac{1}{n}$.]
- *If $\frac{1}{n} T^n f \to 0$ for all $f \in X$, then $A_n f \to 0$ for all $f \in R(I - T)$;*

  [Hint : for $n \in \mathbb{N}$, $(I - T)A_n = A_n(I - T) = \frac{1}{n}(I - T^n)$]
- *If $A_n f \to g$, then $f - g \in \overline{R(I - T)}$.*

Lemma 3.

Let X be a Banach space and $T \in \mathcal{B}(X)$. Then

$$F := \left\{ f \in X : P_T f := \lim_{n \to \infty} A_n f \quad \text{exists} \right\}$$

is a T-invariant subspace of X containing $\text{fix}(T)$. Moreover, $P_T : F \to F$ is a **projection onto** *$\text{fix}(T)$ satisfying $TP_T = P_T T = P_T$ on F.*

Proof.

- The space F is clearly a subspace of X and $P_T : F \to F$ is clearly linear.
- Let $f \in F$. We have $\lim_{n\to\infty} A_n f$ exists and $P_T f := \lim_{n\to\infty} A_n f$. Then $TP_T f = P_T f$, and $\lim_{n\to\infty} A_n Tf = P_T f$. Thus, $P_T Tf = P_T f$.

Definition 4.

Let $T \in \mathcal{B}(X)$. *Consider the operator* P_T *given by*

$$P_T f := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f$$

on the space

$$F := \left\{ f : \lim_{n\to\infty} A_n f \quad \textit{exists} \right\}. \tag{1}$$

The operator P_T *is called the* **mean ergodic projection** *associated with* T. *The operator* T *is called* **mean ergodic** *if* $F = E$, *i.e., the limit in (1) exists for every* $f \in X$.

Using this terminology we can rephrase von Neumann's result : The Koopman operator associated with a measure-preserving system (X, ϕ) is mean ergodic when considered as an operator on $E = L_2(X)$.

Theorem 5.

Let $T \in \mathcal{B}(X)$*. Suppose that* $\sup_{n\in\mathbb{N}} \|A_n\| < \infty$ *and that* $\frac{1}{n}T^n f \to 0$ *for all* $f \in E$*. Then the subspace*

$$F := \left\{ f : \lim_{n\to\infty} A_n f \text{ exists} \right\}$$

is closed, T*-invariant, and decomposes into a direct sum of closed subspaces*

$$F = fix(T) \oplus \overline{R(I-T)} = N(I-T) \oplus \overline{R(I-T)}.$$

The operator $T|_F \in \mathbb{B}(F)$ *is mean ergodic. Furthermore, the operator*

$$P_T : F \to fix(T), \qquad P_T f := \lim_{n\to\infty} A_n f$$

is a bounded projection with $N(P_T) = \overline{R(I-T)}$ *and* $P_T T = P_T = T P_T$*.*

We shall see that the following result gives von Neumann's theorem as a corollary.

Theorem 6 (Theorem 8.6).

Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be a contraction ($\|T\| \leq 1$). Then

$$P_T f := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f \qquad \text{exists for every } f \in H.$$

Moreover, $H = fix(T) \oplus \overline{ran}(I - T)$ is an orthogonal decomposition, and the mean ergodic projection P_T is the orthogonal projection onto $fix(T)$.

The decomposition $H = fix(T) \oplus \overline{ran}(I - T)$ is called the **von Neumann decomposition for contractions**.

Outline of the proof

Since $\|T\| \leq 1$,

$$\|T^n\| \leq \|T\|^n \leq 1$$

and

$$\|A_n[T]\| = \left\| \frac{I + T + T^2 + \cdots + T^{n-1}}{n} \right\| \leq 1,$$

so the powers T^n and $A_n[T]$ are contractions, hence $\sup_{n\in\mathbb{N}} \left\|A_n[T]\right\| < \infty$.

Also $\|\frac{1}{n}T^n\| \leq \frac{1}{n} \to 0$. So $\frac{1}{n}T^n f \to 0$ for all $f \in H$.

Therefore, Theorem 8.5 can be applied and so the subspace F is closed and $P_T : F \to F$ is a projection onto $fix(T)$ with kernel $\overline{ran}(I - T)$.

Note that $F = fix(T) \oplus \overline{ran}(I - T)$.

$H = fix(T) \oplus \overline{ran}(I - T)$

Let $f \in H$ with $f \perp \ ran(I - T)$.

Then $\langle f, f - Tf \rangle = 0$ and hence $\langle f, Tf \rangle = \|f\|^2$.

We have $\|Tf - f\|^2 = \|Tf\|^2 - 2Re\langle f, Tf \rangle + \|f\|^2 = \|Tf\|^2 - \|f\|^2 \leq 0$.

Since T is contraction and $\|Tf - f\|^2 = \|Tf\|^2 - \|f\|^2 \leq 0$, $\|Tf - f\| = 0$, so $f \in \ fix(T)$. Thus $ran(I - T)^{\perp} \subseteq fix(T)$.

Since $fix(T) \cap \overline{ran}(I - T) = \{0\}$, we have

$$ran(I - T)^{\perp} = fix(T).$$

Since P_T is the projection onto $fix(T)$ and $H = fix(T) \oplus \overline{ran}(I - T)$, P_T is the orthogonal projection onto $fix(T)$.

Alternatively, note that P_T must be a contraction and use that a contractive projection is orthogonal.

Corollary 7 (Corollary 8.7).

Let T be a contraction on a Hilbert space H. Then $fix(T) = fix(T^*)$ and $P_T = P_{T^*}$.

Proof :

Let $f \in fix(T^*)$. Then $\langle Tf, f \rangle = \langle f, T^* f \rangle = \|f\|^2$. We have $\|Tf - f\|^2 = \|Tf\|^2 - 2Re\langle f, Tf \rangle + \|f\|^2 = \|Tf\|^2 - \|f\|^2 \leq 0$. Since T is contraction and $\|Tf - f\|^2 = \|Tf\|^2 - \|f\|^2 \leq 0$, $\|Tf - f\| = 0$, so $f \in fix(T)$. By symmetry, $fix(T) = fix(T^*)$.

Since $fix(T) = fix(T^*)$, P_T and P_{T^*} are orthogonal projections onto the same closed subspace of H, hence $P_T = P_{T^*}$.

Alternatively, one may argue as follows. Since $A_n[T] \to P_T$ strongly, i.e., pointwise on H, $A_n[T^*] = A_n[T]^* \to P_T^* = P_T$ weakly. But T^* is a contraction as well, hence $A_n[T^*] \to P_{T^*}$ strongly. Hence $P_T = P_{T^*}$ and $fix(T) = fix(T^*)$.

Exercise 8.

Let $H = \ell_2$. *Define* $T : \ell^2 \to \ell^2$ *by*

$$T(x_1, x_2, x_3, \ldots) = (x_1, x_2/2, x_3/3, \ldots).$$

Anwer the following :

- *Is* T *a bounded operator on* H*?*
- *Is* T *a contraction?*
- *Define* T *suitably on* H *so that* $fix(T) = fix(T^*)$ *and* $P_T = P_{T^*}$.

Exercise

Exercise 9.

Let $H = L_2([0,1])$. *Define* $T : H \to H$ *by*

$$T(x(t)) = x(1-t).$$

Is T *a contraction?*

Operator semigroup

Let X be a Banach space and $\mathbb{T} \subseteq \mathcal{B}(X)$.

Definition 10.

- $\mathbb{T}$ *is called an* **(operator) semigroup** *if*

$$\mathbb{T} \cdot \mathbb{T} := \{ST : S, T \in \mathbb{T}\} \subseteq \mathbb{T}.$$

- *A semigroup* $\mathbb{T}$ *is called* **mean ergodic** *if* $\exists P \in \mathcal{B}(X)$:
  (a) $TP = PT = P \forall T \in \mathbb{T}$ *and*
  (b) $Pf \in \overline{conv}\{\mathbb{T}f\} := \overline{conv}\{Tf : T \in \mathbb{T}\} \forall f \in X$.

We call P *the corresponding* **mean ergodic projection**.

Observations

Since

$$TP = P \Rightarrow Pf \in fix(\mathbb{T}) := \bigcap_{T \in \mathbb{T}} fix(T)$$

and

$$Pf \in \overline{conv}\{\mathbb{T}f\} \Rightarrow P|_{fix(\mathbb{T})} = \text{identity},$$

therefore,

$$P(Pf) = Pf \Rightarrow P^2 = P.$$

Moreover, P is unique.

Theorem 11.

Let $\mathbb{T}$ be a contraction semigroup on a Hilbert space H and P be a projection onto fix $(\mathbb{T})$. Then $\mathbb{T}$ is mean ergodic with associated projection P. Also, Pf is the unique element of $\overline{conv}\{\mathbb{T}f\}$ with minimal norm.

Proof.

By definition, $TP = P$. Since $fix(T) = fix(T^*)$. So, $PT = P$. As $\overline{conv}\{\mathbb{T}f\}$ is non-empty, closed, and convex, $\exists! g \in \overline{conv}\{\mathbb{T}f\}$ with minimal norm. Then $Tg \in \overline{conv}\{\mathbb{T}f\}$ and $\|Tg\| \leq \|g\|$. Thus $Tg = g \Rightarrow g \in fix(\mathbb{T})$.

Hence

$$g = Pg \in \overline{conv}\{PTf : T \in \mathbb{T}\} = \{Pf\}.$$

Bounded mean ergodic semigroups

Theorem 12.

Let $\mathbb{T}$ be bounded (by $c > 0$) semigroup on E. The following are equivalent :

(i) $\overline{conv}\{\mathbb{T}f\} \cap fix(\mathbb{T})$ *is a singleton for each* $f \in E$.

(ii) $\mathbb{T}$ *is mean ergodic.*

(iii) $\overline{conv}\{\mathbb{T}f\} \cap fix(\mathbb{T}) \neq \emptyset \ \forall f \in E$ *and* $\overline{conv}^{w^*}\{\mathbb{T}'f'\} \cap fix(\mathbb{T}') \neq \emptyset \ \forall f' \in E'$.

In this case, $\overline{conv}\{\mathbb{T}f\} \cap fix(\mathbb{T}) = \{Pf\} \ \forall f \in E$.

Let $S \in \mathbb{T}$. Then $Spf = Pf$. Also

$$PSf \in \overline{conv}\{JSf\} \cap fix(\mathbb{T}) \subseteq \overline{conv}\{\mathbb{T}f\} \cap fix(\mathbb{T}) = \{Pf\}.$$

Note that, $\|Pf\| \leq c\|f\|$ and $P(\lambda f) = \lambda Pf$. For $f, g \in E$ and $\varepsilon > 0$, $\exists\ S, R \in \overline{conv}\{\mathbb{T}\}$ with

$$\|Pf - Sf\| \leq \varepsilon \quad \text{and} \quad \|PSg - RSg\| \leq \varepsilon.$$

Then

$$\begin{aligned} \|(Pf + Pg) - RS(f+g)\| &\leq \|RPf - RSf\| + \|Pg - RSg\| \\ &= \|RPf - RSf\| + \|PSg - RSg\| \\ &\leq (c+1)\varepsilon. \end{aligned}$$

So, $Pf + Pg \in \overline{conv}\{T(f+g) : T \in \mathbb{T}\} \cap fix(\mathbb{T}) = \{P(f+g)\}$.

Since $Pf \in \overline{conv}\{\mathbb{T}f\} \cap fix(\mathbb{T})$, so the latter is non-empty.
Next,

$$T'P' = (PT)' = P' \forall T \in \mathbb{T} \Rightarrow P'f' \in fix(\mathbb{T}').$$

Also, $P'f' \in \overline{conv}^{w^*}\{\mathbb{T}'f'\}$ (Hahn-Banach Separation theorem).
Thus

$$\overline{conv}^{w^*}\{\mathbb{T}'f'\} \cap fix(\mathbb{T}') \neq \emptyset.$$

Let $u, v \in \overline{conv}\{\mathbb{T}f\} \cap fix(\mathbb{T})$ and

$$C := \{f' \in E' : \|f'\| \leq 1, \langle u - v, f' \rangle = \|u - v\|\}.$$

Then C is convex, weak*-closed, non-empty (Hahn-Banach), and $\mathbb{T}'$-invariant.
Hence

$$\emptyset \neq \overline{conv}^{w^*}\{\mathbb{T}'f'\} \cap fix(\mathbb{T}') \subseteq C \cap fix(\mathbb{T}').$$

Thus $\exists f' \in C \cap fix(\mathbb{T}')$ and so $\langle f, f' \rangle = \langle Tf, f' \rangle$. Therefore $\|u - v\| = \langle u - v, f' \rangle = \langle u, f' \rangle - \langle v, f') = \langle f, f' \rangle - \langle f, f' \rangle = 0$.
This proves

$$u = v.$$

Reference

Ulrich Krengel, *Ergodic Theorems*, Walter de Gruyter, 1985.